UNIVERSITY COLLEGE LONDON

EXAMINATION FOR INTERNAL STUDENTS

MODULE CODE : **MATH1402**

MODULE NAME : **Mathematical Methods 2**

DATE : **01-May-07**

TIME : **10:00**

TIME ALLOWED : **2 Hours 0 Minutes**

2006/07-MATH1402A-001-EXAM-233

TURN OVER

All questions may be attempted but only marks obtained on the best ***four*** *solutions will count.*
The use of an electronic calculator is **not** *permitted in this examination.*

1. (a) Find the length of the curve C defined by

$$\mathbf{r}(t) = (\exp(t), \exp(t)\cos(t), \exp(t)\sin(t)),$$

for $0 \leq t \leq 2\pi$. Sketch C.

(b) Using the integral methods of this course:

(i) prove that the area of the disk of radius a is πa^2,
(ii) prove that the surface area of the sphere of radius a is $4\pi a^2$,
(iii) prove that the volume of the ball of radius a is $\frac{4}{3}\pi a^3$,
(iv) prove that the volume of an axisymmetric cone of height h and base with radius a is $\frac{1}{3}\pi a^2 h$.

2. (a) State, without proof, the general formula for a Fourier series on $(-\pi, \pi)$ for a function $f(x)$, giving the expressions for the coefficients.

(b) Find the Fourier series of $f(x) = \exp(x)$ on $(-\pi, \pi)$.

(c) Hence, or otherwise, find the Fourier series of $\sinh(x)$ and $\cosh(x)$ on $(-\pi, \pi)$.

3. (a) Using subscript notation, what is the expression for

$$\varepsilon_{ijk}\varepsilon_{klm}$$

in terms of δ_{il}, δ_{jm}, etc.?

(b) Using subscript notation, prove that

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B}),$$

and

$$(\mathbf{A} \times \mathbf{B}) \cdot (\mathbf{C} \times \mathbf{D}) = (\mathbf{A} \cdot \mathbf{C})(\mathbf{B} \cdot \mathbf{D}) - (\mathbf{A} \cdot \mathbf{D})(\mathbf{B} \cdot \mathbf{C}).$$

(c) Hence, or otherwise, prove that

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) + \mathbf{B} \times (\mathbf{C} \times \mathbf{A}) + \mathbf{C} \times (\mathbf{A} \times \mathbf{B}) = 0.$$

4. (a) State Stokes' theorem carefully.

   (b) Using Stokes' theorem, and appropriately cutting the surface into two parts, prove

$$\oint \operatorname{curl} \mathbf{A} \cdot \mathbf{n}\, dS = 0,$$

   where S is a smooth closed surface, and $\mathbf{n}$ is a unit outward normal to S.

   (c) State the divergence theorem carefully.

   (d) Prove the result of part 4b using the divergence theorem, proving any differential identity that you use.

   (e) Verify the result of part 4b by direct evaluation in the case where S is given by $x^2 + y^2 + z^2 = 1$ in $z > 0$ and $x^2 + y^2 \leq 1$ in $z = 0$, and $\mathbf{A} = \mathbf{k} \times \mathbf{r}$, where $\mathbf{r}$ is the position vector and $\mathbf{k}$ is the unit vector in the z direction. Sketch S.

5. (a) State Green's theorem in the plane carefully.

   (b) Verify Green's theorem for the region R defined by $x^2 + y^2 \leq 1$, $x + y \geq 0$ and $x - y \geq 0$, for the functions

$$P(x, y) = xy, \qquad Q(x, y) = x^2,$$

   using the standard notation. Sketch the region R.

6. A physical system is governed by the following equations:

$$\operatorname{div} \mathbf{E} = \rho, \qquad \operatorname{curl} \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \qquad \operatorname{div} \mathbf{B} = 0, \qquad \operatorname{curl} \mathbf{B} = \mathbf{J} + \frac{\partial \mathbf{E}}{\partial t},$$

where t is time. You may assume that time derivatvies commute with ∇.

   (a) Show that

$$\frac{\partial \rho}{\partial t} + \operatorname{div} \mathbf{J} = 0.$$

   (b) When $\rho = 0$ and $\mathbf{J} = 0$ everywhere, show that

$$\nabla^2 \mathbf{E} - \frac{\partial^2 \mathbf{E}}{\partial t^2} = 0, \qquad \text{and} \qquad \nabla^2 \mathbf{B} - \frac{\partial^2 \mathbf{B}}{\partial t^2} = 0.$$